

Homodyne detection of matter-wave fields

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A scheme is discussed that allows one for performing homodyne detection of the matter-wave field of ultracold bosonic atoms. It is based on a pump-probe lasers setup, that both illuminates a Bose-Einstein condensate, acting as reference system, and a second ultracold gas, composed by the same atoms but in a quantum phase to determine. Photon scattering outcouples atoms from both systems, which then propagate freely. Under appropriate conditions, when the same photon can either be scattered by the Bose-Einstein condensate or by the other quantum gas, both flux of outcoupled atoms and scattered photons exhibit oscillations, whose amplitude is proportional to the condensate fraction of the quantum gas. The setup can be extended to measure the first-order correlation function of a quantum gas. The dynamics here discussed make use of the entanglement between atoms and photons which is established by the scattering process in order to access detailed information on the quantum state of matter.

I. INTRODUCTION

The measurement technique which is mostly employed in experiments with ultracold atoms is based on time of flight [1, 2]. Used in different setups it allows one to determine the quantum state of matter [3–5]. Further exciting perspectives have been opened by the recent demonstration of in-situ imaging of atoms in optical lattices [6–9]. In addition, the remarkable progress in coupling ultracold atomic gases with high-finesse optical resonators [10–13] has generated a renewed interest in revealing the correlation functions of matter by photodetection of the scattered light. This progress opens, amongst several others, the possibility to monitor properties of the quantum state of matter in a non-destructive way [14–17].

Photons emitted by Rayleigh scattering, and in general in a pump-probe type of experiment, have been used to characterize the quantum state of atomic gases [18–22]. They deliver information on the structure form factor of the quantum gas and thus on the density and density-density correlations [3, 22–24]. Proposals for optical detection of certain quantum states of matter have appeared in the literature [23–27].

In this article we discuss a setup in which detection of the photons scattered by a quantum gas permits one to determine the mean value of the atomic field. This setup constitutes a matter-wave analogon of homodyne detection of light fields [28], and specifically allows one to determine the condensate fraction of a quantum gas [29] by photodetection. Furthermore we show that this scheme can be extended to determine the first-order correlation function of an atomic gas when the two illuminated regions belong to the same quantum gas.

The setup for determining the condensate fraction is sketched in Fig. 1 and is based on an interferometer for Bose-Einstein Condensates (BEC) realized in Refs. [30, 31]. We establish a direct connection between our theoretical model and the experimental setup in Ref. [30, 31] and then focus on a scheme in which the

second system is not necessarily a BEC. We then show that this setup allows one to determine the condensate fraction of the second system by measuring the flux of the scattered photon. The dynamics is based on first creating entanglement between the scattered photon and the scattering atom of the quantum gas and then on performing an operation similar to a quantum eraser [32], thereby allowing one to measure the correlation functions of matter in the flux of the scattered light.

This article is organized as follows. In Sec. II the setup and the theoretical model are introduced. In Sec. III a formal connection is derived between the photon flux and the correlation functions of the scattering atoms. In Sec. IV the photon flux is evaluated when the scattering systems are two Bose-Einstein condensates at zero temperature and at different temperatures. In Sec. V it is discussed how a similar setup can be used to measure the first-order correlation function. The conclusions are drawn in Sec. VI. The appendices report details of the calculations presented in Secs. II, III, and IV.

II. THE MODEL

A realization of the setup we consider in this paper is depicted in Fig. 1. Here, ultracold bosonic atoms of mass m are initially prepared in the stable electronic state $|1\rangle$ and confined by the (state-dependent) potential $V_1(\mathbf{r})$. The atoms that are illuminated by the lasers are coherently coupled to the electronic state $|2\rangle$, which is stable and not confined: the outcoupled atoms propagate freely in space.

In this section we introduce the Hamiltonian and the physical quantities which are at the basis of our analysis.

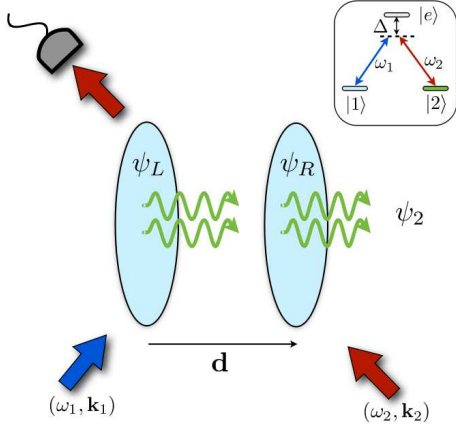


FIG. 1: Setup for performing homodyne detection of matter-wave fields. A gas of identical bosonic atoms is confined in two distinguishable regions of space. In the left region it forms a Bose-Einstein condensate that acts as reference system for measuring the condensate fraction of the right system. A pump-probe laser setup outcouples the atoms from both wells. The photon flux of the probe beam is constituted by the photon scattered in the outcoupling process and exhibits oscillations whose amplitude depend on the condensate fraction of the right system, and which vanish when this is zero. The first-order correlation function of a quantum gas can be determined in an extension of this setup and is discussed in Sec. V. Details on the parameters are given in the text.

A. Hamiltonian

We denote by H the Hamiltonian governing the dynamics of photons and atoms, which we decompose into the sum

$$H = H_0 + H_{\text{emf}} + H_{\text{int}} + H_{\text{shift}}, \quad (1)$$

with H_0 the Hamiltonian governing the atoms dynamics in absence of interaction with the lasers, H_{emf} the Hamiltonian for the free transverse electromagnetic (e.m.) field, while H_{int} describes the coupling between states $|1\rangle$ and $|2\rangle$ induced by coherent Raman scattering. Hamiltonian H_{shift} includes the dynamical Stark shift due to off-resonant coupling between atoms and e.m.-field modes. These Hamiltonians are given using a second-quantized description of matter- and photon-fields. In detail

$$H_{\text{emf}} = \sum_{\lambda} \hbar \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda}$$

gives the energy of the transverse electromagnetic field in free space (without the vacuum energy) where λ labels a mode of the e.m.-field at wave vector \mathbf{k} , polarization $\vec{\epsilon} \perp \mathbf{k}$, and frequency $\omega_{\mathbf{k}} = c|\mathbf{k}|$, with c the velocity of light. The operators a_{λ} and a_{λ}^{\dagger} annihilate and create, respectively, a photon in mode λ , and obey the bosonic commutation relation $[a_{\lambda}, a_{\lambda'}^{\dagger}] = \delta_{\lambda, \lambda'}$.

The atomic Hamiltonian H_0 is conveniently rewritten

as $H_0 = H_1 + H_2 + H_{12}$, where

$$H_{\{j=1,2\}} = \int d\mathbf{r} \psi_j^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} + V_j(\mathbf{r}) + \hbar \omega_{aj} \right) \psi_j(\mathbf{r}) + \frac{g_j}{2} \int d\mathbf{r} \psi_j^{\dagger}(\mathbf{r}) \psi_j^{\dagger}(\mathbf{r}) \psi_j(\mathbf{r}) \psi_j(\mathbf{r}) \quad (2)$$

describes the dynamics of the atoms in the electronic state $|j\rangle$ at frequency ω_{aj} . Here $\psi_j(\mathbf{r})$ and $\psi_j^{\dagger}(\mathbf{r})$ are the annihilation and creation operator, respectively for a bosonic atom at position \mathbf{r} and in the electronic state $|j = 1, 2\rangle$, obeying the commutation relation $[\psi_j(\mathbf{r}), \psi_k^{\dagger}(\mathbf{r}')] = \delta_{j,k} \delta(\mathbf{r} - \mathbf{r}')$. Parameter $g_j = 4\pi \hbar^2 a_{s,j}/m$ is the strength of s -wave scattering between the atoms in state $|j\rangle$, with $a_{s,j}$ the corresponding s -wave scattering length[29]. Hamiltonian term H_{12} describes s -wave scattering between an atom in electronic state $|1\rangle$ and an atom in electronic state $|2\rangle$, see [33, 34], and will be negligible in the situations we consider.

The Hamiltonian terms for the interactions between photons and atoms are given using normal ordering [35]. They include a term describing coherent Raman coupling between the states $|1\rangle$ and $|2\rangle$, in which a photon in mode 1 is scattered into mode 2 and vice versa. Raman transitions follow from a pump-probe type of excitation and the corresponding Hamiltonian reads

$$H_{\text{int}} = \hbar \int d\mathbf{r} \left[\gamma(\mathbf{r}) a_2^{\dagger} \psi_2^{\dagger}(\mathbf{r}) \psi_1(\mathbf{r}) a_1 + \text{H.c.} \right], \quad (3)$$

with the position-dependent coupling strength $\gamma(\mathbf{r})$ [36]. For later convenience we denote by the quantity

$$\omega_{12} = \omega_1 - \omega_2$$

the difference between the frequencies of the two e.m.f.-modes. Moreover, we assume $\omega_{12} \sim \omega_{a2} - \omega_{a1}$, i.e., the two lasers drive quasi-resonantly the Raman transition coupling the stable electronic states $|1\rangle$ and $|2\rangle$. The interaction with the pump and probe laser also induces a dynamical Stark shift, whose corresponding Hamiltonian reads (in normally-ordered form)

$$H_{\text{shift}} = \sum_{j=1}^2 \hbar \int d\mathbf{r} \gamma_j(\mathbf{r}) a_j^{\dagger} \psi_j^{\dagger}(\mathbf{r}) \psi_j(\mathbf{r}) a_j, \quad (4)$$

where the parameter $\gamma_j(\mathbf{r})$ has the dimension of an angular frequency. It can be written as $\gamma_j(\mathbf{r}) = 2|\Omega_j(\mathbf{r})|^2/\Delta$, with $\Omega_j(\mathbf{r})$ the Rabi frequency of the mode coupling the transition $|j\rangle \rightarrow |e\rangle$ and Δ the detuning between the mode and the atomic transition frequencies (the Lamb shifts are included in the frequencies of the atomic states).

In the following we will assume that the pump and probe fields are traveling waves with wave vectors \mathbf{k}_j . In this case, the Raman-coupling strength $\gamma(\mathbf{r})$ reads

$$\gamma(\mathbf{r}) = \gamma_0 e^{i\mathbf{q} \cdot \mathbf{r}} \quad (5)$$

with $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$ and $\gamma_0 = 2\Omega_1(\mathbf{r})\Omega_2(\mathbf{r})^*/\Delta$.

We remark that the outcoupling of atoms from two Bose-Einstein condensates in the setup of Refs. [30, 31] was performed by Bragg scattering while we model the outcoupling process by Raman coupling to a different hyperfine state. This difference however does not affect the general results, as it will be shown in the following

B. Spin-dependent potential

We will now provide more details on the potential which confines the atoms. We will assume that the atoms in state $|2\rangle$ propagate freely, namely, potential $V_2(\mathbf{r})$ in Eq. (2) is a constant which we set equal to zero.

The atoms in state $|1\rangle$ are trapped by potential $V_1(\mathbf{r})$. This potential confines the atoms in two spatially separated regions around the two potential minima, which are at distance d and, specifically, are localized at the points $\mathbf{r}_L = -(d/2)\hat{x}$ and $\mathbf{r}_R = (d/2)\hat{x}$. The potential can be then decomposed in the sum $V_1(\mathbf{r}) = V_L(\mathbf{r}) + V_R(\mathbf{r})$, with $V_j(\mathbf{r})$ the potential centered at \mathbf{r}_j and $j = L, R$. The two atomic clouds at each well are initially uncorrelated and there is no tunneling between the two spatial region. In principle, hence, the atoms in the two regions of space are distinguishable.

It is useful to consider the partition $\psi_1(\mathbf{r}) = \psi_L(\mathbf{r}) + \psi_R(\mathbf{r})$ where we denote by $\psi_L(\mathbf{r}, t) = \psi_1(\mathbf{r}, t)\theta(-z)$ and $\psi_R(\mathbf{r}, t) = \psi_1(\mathbf{r}, t)\theta(z)$ the field operators which do not vanish on the left and right region of space, respectively, such that $[\psi_j(\mathbf{r}), \psi_k^\dagger(\mathbf{r}')] = \delta_{j,k}\delta(\mathbf{r} - \mathbf{r}')$ for $\mathbf{r}, \mathbf{r}' \neq 0$ [37]. Using these definitions we can write $H_1 = H_L + H_R$ with

$$H_{\{j=L,R\}} = \int d\mathbf{r} \psi_j^\dagger(\mathbf{r}) \left(\frac{-\hbar^2 \nabla^2}{2m} + V_j(\mathbf{r}) \right) \psi_j(\mathbf{r}) + \frac{g}{2} \int d\mathbf{r} \psi_j^\dagger(\mathbf{r}) \psi_j^\dagger(\mathbf{r}) \psi_j(\mathbf{r}) \psi_j(\mathbf{r}), \quad (6)$$

and where we have set $\omega_{a1} = 0$ and $g_1 = g$. In this representation the Hamiltonian describing the interaction with the lasers takes the form

$$H_{\text{int}} = \hbar \int d\mathbf{r} \left[\gamma(\mathbf{r}) a_2^\dagger \psi_2^\dagger(\mathbf{r}) (\psi_L(\mathbf{r}) + \psi_R(\mathbf{r})) a_1 + \text{H.c.} \right]. \quad (7)$$

We note that our model is an extension of the one in Refs. [38, 39] where the authors studied the outcoupling of atoms from a single BEC by means of *classical* Raman lasers. By considering the quantum dynamics of the interaction between photons and atoms we take into account the quantum fluctuations of the light field due to the scattering process, which are instead discarded in [38, 39]. We will indeed show that these fluctuations give access to some correlation functions of the scattering system.

C. The scattered field

We now study the properties of the scattered photons by considering the photon field operators in Heisenberg picture. The Heisenberg equations of motion are determined assuming that the coupling between matter and photons is sufficiently weak to be treated in second-order perturbation theory. In this limit the operators for the photonic modes read

$$a_1(t) = a_1(0)e^{-i(\omega_1 t + \phi_1(t))} - i e^{-i\omega_1 t} \int_0^t d\tau e^{i\omega_{12}\tau} \int d\mathbf{r} \gamma(\mathbf{r}) \left[\psi_L^{(0)\dagger}(\mathbf{r}, \tau) + \psi_R^{(0)\dagger}(\mathbf{r}, \tau) \right] \psi_2^{(0)}(\mathbf{r}, \tau) a_2(0), \quad (8)$$

$$a_2(t) = a_2(0)e^{-i(\omega_2 t + \phi_2(t))} - i e^{-i\omega_2 t} \int_0^t d\tau e^{-i\omega_{12}\tau} \int d\mathbf{r} \gamma(\mathbf{r}) \psi_2^{(0)\dagger}(\mathbf{r}, \tau) \left[\psi_L^{(0)}(\mathbf{r}, \tau) + \psi_R^{(0)}(\mathbf{r}, \tau) \right] a_1(0), \quad (9)$$

where $\psi_j^{(0)}(\mathbf{r}, \tau) = \exp(iH_0\tau/\hbar)\psi_j(\mathbf{r}, 0)\exp(-iH_0\tau/\hbar)$. The physical origin of the individual terms on the Right-Hand Side (RHS) can be simply identified. The first term on the RHS of both equations is the free-field component. It is characterised by a time-dependent phase $\phi_j(t)$, which is proportional to the atomic density in the electronic state $|j\rangle$. When the Born approximation can be performed, it is a density-dependent phase shift of the

field mode, with the form

$$\phi_1(t) = \sum_{j,k=L,R} \int_0^t d\tau \int d\mathbf{r} \gamma_1(\mathbf{r}) \psi_j^{(0)\dagger}(\mathbf{r}, \tau) \psi_k^{(0)}(\mathbf{r}, \tau),$$

$$\phi_2(t) = \int_0^t d\tau \int d\mathbf{r} \gamma_2(\mathbf{r}) \psi_2^{(0)\dagger}(\mathbf{r}, \tau) \psi_2^{(0)}(\mathbf{r}, \tau).$$

Given that all atoms are initially prepared in the electronic state $|1\rangle$, term $\phi_1(t)$ gives rise to a frequency shift of field-mode 1. This shift is proportional to the density of the medium integrated over the path along which light propagates [3, 40].

The second term on the RHS establishes a direct proportionality relation between the photonic and the matter-wave fields a_2 and $\psi_1(\mathbf{r})$ (a_1 and $\psi_2(\mathbf{r})$). It shows, in particular, that the source term of field a_1 is the coherent overlap of the field scattered from the right and from the left wells. In this shape the outcoupling process in this system resembles a beam-splitter operation. This property establishes an analogy with an interferometric setup, which we will exploit in order to perform homodyne detection of matter-wave field. Differing from a simple interferometer, however, atoms and photons are correlated by the scattering process. This important difference is at the basis of the dynamics we observe.

Before we discuss the signal at the photodetector, we will introduce a few approximations which will notably simplify the treatment. In first place we neglect atomic collisions between outcoupled atoms in state $|2\rangle$, since we assume that the gas of outcoupled atoms is at very low densities. This regime also permits us to neglect collisions between outcoupled atoms and trapped atoms in state $|1\rangle$ [41].

III. PHOTON FLUX

The quantity we analyse in this paper is the flux of photons of mode 2, namely, the rate of change of the photon number in the mode at frequency ω_2 . Formally, the photon flux is given by the equation

$$F(t) = \frac{d}{dt} \langle a_2^\dagger(t) a_2(t) \rangle \quad (10)$$

where the expectation value $\langle \cdot \rangle$ is taken over the initial state of the atoms and the e.m.-field. The photon flux, integrated over the detection time, gives the integrated intensity of the field at the detector. It can be verified that

$$F(t) \propto \frac{d}{dt} \int d\mathbf{r} \langle \psi_2^\dagger(\mathbf{r}, t) \psi_2(\mathbf{r}, t) \rangle \quad (11)$$

where the proportionality factor is the mean number of photons in mode 1. This equality shows indeed that flux of scattered photons and of the corresponding outcoupled atoms carry the same information.

For the specific setup we consider, the photon flux can be rewritten as the sum of two contributions,

$$F = F_B + F_I, \quad (12)$$

with

$$F_B(t) = \Gamma \text{Re} \sum_{j=R,L} \int d\mathbf{r} \int d\mathbf{r}' \times \int_0^t dt' f(\mathbf{r}, t; \mathbf{r}', t') G_{jj}(\mathbf{r}, t; \mathbf{r}', t'), \quad (13a)$$

$$F_I(t) = \Gamma \text{Re} \int d\mathbf{r} \int d\mathbf{r}' \int_0^t dt' f(\mathbf{r}, t; \mathbf{r}', t') \times (G_{LR}(\mathbf{r}, t; \mathbf{r}', t') + G_{RL}(\mathbf{r}, t; \mathbf{r}', t')) \quad (13b)$$

where $\Gamma = 2\langle a_1^\dagger a_1 \rangle \gamma_0^2$ is a scaling factor that is proportional to the number of photons in mode 1. Here,

$$G_{jk}(\mathbf{r}, t; \mathbf{r}', t') = \langle \psi_j^\dagger(\mathbf{r}', t') \psi_k(\mathbf{r}, t) \rangle, \quad (14)$$

$$f(\mathbf{r}, t; \mathbf{r}', t') = e^{i[\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}') - \omega_{12}(t - t')]} \langle \psi_2(\mathbf{r}', t') \psi_2^\dagger(\mathbf{r}, t) \rangle, \quad (15)$$

where the initial state is the vacuum state of the e.m. field except for mode 1 and 2, which are assumed to be in coherent states with non-vanishing photon number, while all atoms are in internal state $|1\rangle$ and at equilibrium in the grand-canonical ensemble at temperature T . The atoms in state $|2\rangle$ propagate freely and do not undergo collisions since we assume that the density is very low. Therefore, Eq. (15) can be cast in the form

$$f(\mathbf{r}, t; \mathbf{r}', t') = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i[(\mathbf{q} - \mathbf{k}) \cdot (\mathbf{r} - \mathbf{r}') - (\tilde{\omega}_{12} - \omega_{\mathbf{k}})(t - t')]}, \quad (16)$$

with

$$\omega_{\mathbf{k}} = \frac{\hbar \mathbf{k}^2}{2m}$$

the recoil frequency and

$$\tilde{\omega}_{12} = \omega_{12} - \omega_{a2}$$

the two-photon detuning.

In order to obtain explicit expressions for the correlation functions $G_{jk}(\mathbf{r}, t; \mathbf{r}', t')$ it is convenient to work in the interaction picture with respect to the grand canonical ensemble $K_0 = H_0 - \sum_{j=L,R} \mu_j \mathcal{N}_j$, where μ_j is the chemical potential of the atoms in either the right or left cloud and $\mathcal{N}_j = \int d\mathbf{r} \psi_j^\dagger(\mathbf{r}) \psi_j(\mathbf{r})$. The new atomic field operators are obtained from the ones in Heisenberg picture with respect to H_0 by replacing $\psi_j(\mathbf{r}, t) \rightarrow \psi_j(\mathbf{r}, t) e^{-\frac{i}{\hbar} \mu_j t}$. From now on we will denote by $\psi_j(\mathbf{r}, t)$ the atomic field operators in Heisenberg picture with respect to K_0 .

We now discuss how the considered setup can be used to measure the mean value of the atomic field operator by means of photons. We first note that the component F_B of the photon flux is the sum of the flux from each well, while the component F_I arises from the coherent superposition of a photon (atom) scattered by the wells. We will denote F_B by “background contribution” and F_I by “interference contribution”. In absence of initial correlations, this latter term is proportional to the product of the mean value of the field operators $\langle\psi_L(\mathbf{r},t)\rangle\langle\psi_R(\mathbf{r},t)\rangle^*$. Let us now consider the case in which, say, the left well confines weakly-interacting atoms forming a Bose-Einstein condensate. Under this assumption the atomic field operator can be written as [29]

$$\psi_L(\mathbf{r},t) = e^{-\frac{i}{\hbar}\mu_L t} (\Phi_L(\mathbf{r}) + \delta\psi_L(\mathbf{r},t)), \quad (17)$$

where $\Phi_L(\mathbf{r})$ is the macroscopic wave function which solves the Gross-Pitaevskii equation for the quantum gas in the left potential well (with chemical potential μ_L). Then, when the order parameter of the left condensate is known, the interference term will deliver the mean value of the field operator in the right well.

In the following we use Eq. (17) in the equations for the background contribution to the photon flux, Eq. (13a), and for the interference term, Eq. (13b). In particular we will take

$$\Phi_L(\mathbf{r}) = \langle\psi_j(\mathbf{r},t)\rangle = f_L(\mathbf{r})e^{i\varphi_L} \quad (18)$$

and assume that both $f_L(\mathbf{r})$ and φ_L are real valued. Hence, $f_L(\mathbf{r})^2$ is the density of condensed atoms, while the phase φ_L is assumed to be constant in space (hence discarding the possibility of superfluid currents in the Bose-Einstein condensate [29]).

A. Background contribution

The background contribution can be decomposed into the sum of the photon flux from the left and from the right well, $F_B = F_L + F_R$. An explicit form of F_L can be provided since we make a specific assumption on the quantum state of the gas trapped in the left well. For this purpose, we consider the integrand $G_{LL}(\mathbf{r},t;\mathbf{r}',t')$ in F_L and observe that it can be splitted into two terms:

$$G_{LL}(\mathbf{r},t;\mathbf{r}',t') = e^{i\mu_L(t-t')/\hbar} [f_L(\mathbf{r})^2 + \delta G_{LL}(\mathbf{r},t;\mathbf{r}',t')] , \quad (19)$$

where

$$\begin{aligned} \delta G_{LL}(\mathbf{r},t;\mathbf{r}',t') &= \langle\delta\psi_L^\dagger(\mathbf{r},t)\delta\psi_L(\mathbf{r}',t')\rangle \\ &= \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} N_0(\omega) A_{\delta\psi_L\delta\psi_L^\dagger}(\mathbf{r},\mathbf{r}',-\omega) \end{aligned} \quad (20)$$

accounts for the contribution of the noncondensed atoms to the photon flux. Here,

$$A_{\delta\psi\delta\psi^\dagger}(\mathbf{r},\mathbf{r}',\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle[\delta\psi(\mathbf{r},\tau), \delta\psi^\dagger(\mathbf{r}',0)]\rangle \quad (21)$$

is the spectral density [42], with $N_0(\omega) = [e^{\beta\hbar\omega} - 1]^{-1}$. Similar expressions can be found for the background contribution of the quantum gas in the right well.

Insight on these equations can be gained by assuming that the gas is homogeneous. This assumption allows us to write the spectral density as $A_{\delta\psi_j\delta\psi_j^\dagger}(\mathbf{r},\mathbf{r}',\omega) = A_{\delta\psi_j\delta\psi_j^\dagger}(|\mathbf{r}-\mathbf{r}'|,\omega)$. We then inspect the equations for the flux using the momentum representation for the atomic field operator. The terms giving the contribution to the photon flux from the left cloud can be cast in the form [43]

$$F_L(t) = 2\pi\Gamma\text{Re} \int \frac{d\mathbf{k}}{(2\pi)^3} |\tilde{f}_L(\mathbf{k})|^2 \tilde{\delta}^t(\Omega - \omega_{\mathbf{k}+\mathbf{q}}), \quad (22a)$$

$$\begin{aligned} \delta F_L(t) &= 2\pi\Gamma V \text{Re} \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} \\ &\times N_0(\omega) \tilde{A}_{\delta\psi_L\delta\psi_L^\dagger}(\mathbf{k},-\omega) \tilde{\delta}^t(\omega + \Omega - \omega_{\mathbf{k}+\mathbf{q}}), \end{aligned} \quad (22b)$$

where $\tilde{A}_{\delta\psi_R\delta\psi_R^\dagger}(\mathbf{k},\omega)$ is the spectral weight function, which is the Fourier transform of $A_{\delta\psi_L\delta\psi_L^\dagger}(\mathbf{r},\omega)$ and gives the strength of the collective excitations at wavevector \mathbf{k} and frequency ω , and is weighted by the Bose function $N_0(\omega)$ [42]. The other parameters are the volume V in which the atoms of the left well are confined, the Fourier transform of the macroscopic wave function $\tilde{f}_L(\mathbf{k}) = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} f_L(\mathbf{r})$, and the frequency

$$\Omega = \tilde{\omega}_{12} - \frac{\mu_L}{\hbar}, \quad (23)$$

giving the effective detuning of the laser from the collective transition. Moreover, in Eqs. (22) we have introduced the quantity

$$\tilde{\delta}^t(\omega) = e^{-i\omega t/2} \delta^t(\omega) \quad (24)$$

that is proportional to the diffraction function $\delta^t(\omega) = \sin(\omega t/2)/(\pi\omega)$, enforcing energy conservation for long interaction times t . In particular, $\tilde{\delta}^t(\omega) \rightarrow \delta(\omega)$ for $t \rightarrow \infty$, with $\delta(\omega)$ Dirac-delta function [44]. Definition (24) contains a time-dependent phase and highlights that the corresponding factor tends to unity in the limit in which energy conservation applies. We will keep this time-dependent phase in the equations for the photon flux since we will also consider intermediate times.

The integrals in Eqs. (22) runs over all values of the atomic momentum \mathbf{k} . The integrands are the product of the momentum distribution at $\mathbf{k}' = \mathbf{q} - \mathbf{k}$ and of the diffraction function: The first accounts for the effect of the photon recoil $\hbar\mathbf{q}$ on the atomic distribution due to the outcoupling process while the diffraction

function imposes energy conservation in the scattering process. Equations (22a) and (22b) thus show that the contributions to the flux come from the total number of condensed and noncondensed atoms, respectively, which fulfill energy and momentum conservation of the scattering process.

We remark that according to Eq. (11) the flux evaluated in Eqs. (22) gives also the corresponding component of the atomic flux. The latter agrees with the corresponding expressions derived in Ref. [38] for the atomic flux which is outcoupled from a single BEC by classical fields.

B. Interference contribution

In order to determine the interference contribution to the photon flux, $F_I(t)$ in Eq. (13b), one needs the explicit form of the correlation functions $G_{LR}(\mathbf{r}, t; \mathbf{r}', t')$, $G_{RL}(\mathbf{r}, t; \mathbf{r}', t')$. For the considered setup, however, one can already make general statements. In absence of initial correlations, in fact, they are the product of the mean value of the field operators in each well, and thus take the form

$$\begin{aligned} G_{LR}(\mathbf{r}, t; \mathbf{r}', t') &= e^{i(\mu_L t - \mu_R t')/\hbar} \langle \psi_L^\dagger(\mathbf{r}, t) \rangle \langle \psi_R(\mathbf{r}', t') \rangle \\ &= e^{i(\mu_L t - \mu_R t')/\hbar} e^{-i\varphi_L} f_L(\mathbf{r}) \langle \psi_R(\mathbf{r}', t') \rangle \end{aligned} \quad (25)$$

while $G_{RL}(\mathbf{r}, t; \mathbf{r}', t') = [G_{LR}(\mathbf{r}', t'; \mathbf{r}, t)]^*$.

Equation (25) shows, as it is expected, that these correlation functions are proportional to the mean value of atomic field operator $\langle \psi_R(\mathbf{r}, t) \rangle$. When this is zero the interference contribution vanishes. Otherwise, the amplitude of this term is proportional to the condensate fraction of the quantum gas in the right well. We remark that the noncondensed atoms in the left well do not contribute to the signal because (i) there are no initial correlations between the atoms in the left and right wells and (ii) the mean value $\langle \delta\psi_L(\mathbf{r}, t) \rangle = 0$.

In the shape of Eq. (25) the analogy with homodyne detection, as it is performed with light fields, can be drawn. The condensate in the left well here plays the role of the local oscillator [28]. Identifying the analog of the phase of the local oscillator is, however, a more delicate issue that deserves some more analysis. For this purpose we first assume that the mean value $\langle \psi_R(\mathbf{r}, t) \rangle$ can be evaluated within a mean-field approach, such that $\langle \psi_R(\mathbf{r}, t) \rangle = f_R(\mathbf{r})e^{i\varphi_R}$, with $f_R(\mathbf{r})$ real-valued function and φ_R real constant. The integral in Eq. (13b) can then be cast in the form

$$F_I(t) = 2\pi\Gamma\text{Re} \int \frac{d\mathbf{k}}{(2\pi)^3} \left[e^{i(\delta\mu t - \varphi_{LR})} \tilde{f}_L(\mathbf{k})^* \tilde{f}_R(\mathbf{k}) \tilde{\delta}^t(\Omega - \omega_{\mathbf{k}+\mathbf{q}}) + e^{-i(\delta\mu t - \varphi_{LR})} \tilde{f}_R(\mathbf{k})^* \tilde{f}_L(\mathbf{k}) \tilde{\delta}^t(\Omega - \delta\mu - \omega_{\mathbf{k}+\mathbf{q}}) \right], \quad (26)$$

with

$$\delta\mu = (\mu_L - \mu_R)/\hbar, \quad (27)$$

and

$$\varphi_{LR} = \varphi_L - \varphi_R. \quad (28)$$

Equation (26) indicates that the interference contribution is an oscillating signal. The phase of the oscillation, however, depends on time and oscillates with a frequency determined by the difference $\delta\mu$ between the chemical potentials while the phase offset is determined by the relative phase between the two (quasi-)condensates. This relative phase will be defined only for a single experimental run, being the two quantum gases independent [3, 45–50].

C. Discussion

1. Which-way information and quantum erasers

Some remarks on the oscillating behaviour of the interference term are now in order. For the example considered in Eq. (26) the photon flux oscillates in time with frequency $\delta\mu$. This oscillation is observed when one chooses a time-window at the detector Δt such that $\delta\mu\Delta t \ll 1$. The interference arises from the overlap of the atomic beams from both condensates. It is analogous to the interference between two lasers at different frequencies: interference fringes are observed provided that the time window of the detector is sufficiently small so not to resolve the detuning between the lasers [51, 52]. As in the case of two independent lasers, there is no well defined phase offset: the relative phase is defined only for a single experimental run while the average over a statistically significant number of runs gives no interference pattern [3, 45, 46, 48].

Differing from the situation of two laser beams [51], however, photon scattering here creates correlations among the scattered photon, the corresponding outcoupled atom, and the quantum gas. This correlation is a form of a which-way information, that in general washes out any interference in the photon flux [53] and can be considered as a form of photon-atoms entanglement [54]. Oscillations in the photon flux, and hence interference, can be recovered if the following two conditions are fulfilled: (i) the beam of atoms outcoupled from one well overlap spatially with the wave function of the atoms in the second trap and (ii) if there are either initial correlations between the atoms in the two wells or if they both possess a non-vanishing condensate fraction.

In order to better understand these conditions we first notice that the outcoupled atoms are distinguishable from the trapped atoms since they are in different quantum states. Nevertheless, the Raman beams can, in principle, outcouple one atom from one trap and then load the same atom into the second trap. This effective coupling is at the basis of the analogy with a Josephson Junction that was drawn in Refs. [30, 31], and is the essential element leading to interference. In these terms, it corresponds to a quantum eraser [32]: when it is fulfilled, in fact, the outcoupled atom becomes disentangled with the scattered photon and with the scattering system. Condition (ii) consists in assuming that there exist classical spatial correlations between the wells. It is important since we assume that at $t = 0$ there are no quantum correlations between the two scattering systems [55].

2. Onset of the time-dependent oscillations

We now analyse the onset of oscillations in time. We have argued that the photon flux starts oscillating after a finite time has elapsed from the beginning of the experiment. [30, 31]. This can be also seen in our theory by performing the integral over \mathbf{k} in Eq. (26). In App. A we show that F_I can be recast in the form $F_I(t) = F_{L \rightarrow R}(t) + F_{R \rightarrow L}(t)$, with

$$F_{L \rightarrow R}(t) \approx \Gamma \text{Re} e^{i(\delta\mu t - \varphi_{LR})} \int_0^t d\tau e^{i(\omega_{\mathbf{q}} - \Omega)\tau} h_{LR}(\mathbf{d}, \tau) \quad (29a)$$

$$F_{R \rightarrow L}(t) \approx \Gamma \text{Re} e^{-i(\delta\mu t - \varphi_{LR})} \int_0^t d\tau e^{i(\omega_{\mathbf{q}} - \Omega + \delta\mu)\tau} h_{RL}(-\mathbf{d}, \tau). \quad (29b)$$

Here,

$$h_{jl}(\mathbf{d}, \tau) = \int d\mathbf{r} f_j(\mathbf{r} + \mathbf{d} - v_{\mathbf{q}}\tau) f_l(\mathbf{r}) \quad (30)$$

is the overlap between the left and the right condensate, with one being displaced by the amount $v_{\mathbf{q}}\tau - \mathbf{d}$. Here,

we introduced the recoil velocity

$$v_{\mathbf{q}} = \frac{\hbar \mathbf{q}}{m} \quad (31)$$

that is acquired by the outcoupled atom by scattering the photon. This overlap vanishes at time $\tau = 0$, i.e., when the outcoupling lasers are switched on (recall that the two clouds initially do not overlap). We note that the overlap integral is zero at all times if \mathbf{d} and \mathbf{q} are orthogonal. It may be finite and reach a maximum after a certain time when \mathbf{d} and \mathbf{q} are parallel, say, pointing along the positive x axis. In this case the component $F_{L \rightarrow R}(t)$ may not vanish and can be interpreted as the contribution to the interference flux $F_I(t)$ from the outcoupled atoms that propagate from left to right (for the term $F_{R \rightarrow L}(t)$ this is just opposite). Interference will be observed provided that a sufficiently long time has elapsed to warrant overlap. This corresponds to times $t > t_c$ with

$$t_c \sim \frac{d_{\text{eff}}}{v_q},$$

where $d_{\text{eff}} = d - (\xi_L + \xi_R)/2$ is the effective distance of the two systems taking into account their width ξ_j . In other words for times $t > t_c$ the which-way information has been erased and oscillations in the photon flux can be observed.

In interferometric setups the amount of visibility and which-way information are related by an inequality [53]. It is therefore useful to determine a visibility of the oscillating signal because it contains information on the properties of the scattering systems. A visibility can be defined for sufficiently long times by averaging over several oscillating periods after the instant t_c . In this case it will be proportional to the amplitude of the oscillations of the photon flux, hence, to the product of the condensate fractions of both systems. We remark, once again, that oscillations can be observed only in a single experimental run while they will disappear after performing an ensemble average. Therefore, this behaviour can only be measured in systems whose properties are not deeply modified by the outcoupling of atoms. When this is not verified, the presence of a condensed fraction in the right system can be revealed by performing an ensemble average over a sufficient large number of experiments in which the signal is monitored for sufficiently short times, warranting that the properties of the quantum gas have not been significantly modified, and taking the statistical distribution of the intensity of the photon flux at a given instant of time. An amplitude can be extracted from the width of this distribution by taking into account the finite width of the diffraction function.

IV. HOMODYNE DETECTION OF A QUANTUM GAS

The theory presented so far will be now applied to some specific examples. The idea is to use the setup in

Fig. 1 to determine the mean value of the field operator of a quantum gas by using a Bose-Einstein condensate at known temperature as reference system. Such setup is a matter-wave analog of homodyne detection in quantum optics. The individual elements can be so identified: the BEC acts as a local oscillator, the outcoupling procedure as beam splitter, the relative phase can be varied by changing the interwell distance. The information on the atomic gas is carried by both scattered photons and outcoupled atoms: homodyne detection of the scattered field, hence, allows one to determine the mean value of the quadrature of the atomic field operator.

A. Interference between two Bose-Einstein Condensates at $T = 0$

We first discuss the case in which two BEC are trapped in the left and right well, respectively, and are both illuminated by the pump and probe beams. The outcoupled atoms from both condensates propagate along the direction determined by the vector joining the minima of both wells. The two atomic beams spatially overlap after the time t_c has elapsed. The scattered photons are revealed at a detector in the far-field. This setup has been realized in the experiment presented in Refs. [30, 31] where time-dependent oscillations in the atom and photon flux were measured. We remark that the outcoupling process in the setup of Refs. [30, 31] was performed by Bragg scattering while we model it by Raman scattering into a different hyperfine state. This difference does not affect the results: While in Refs. [30, 31] the atoms are transferred into a momentum state that freely propagates, here the atoms are transferred into a hyperfine state that is not trapped. In both cases, collisions between the outcoupled beam and the trapped atoms can be neglected.

We assume two BEC with equal number of atoms N_C and temperature T , that are confined either in the left or right well (The assumption of equal number of atoms

is a convenient theoretical choice, but no substantial restriction to the results presented in the following). The wells are described by the potential

$$V_{\{j=L,R\}}(\mathbf{r}) = V(\mathbf{r} - \mathbf{r}_j) + \delta_{j,L}\Delta V \quad (32)$$

with $V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, and ΔV denotes the constant offset between the two traps. For simplicity we take that both BEC are at zero temperature, $T = 0$, and the atoms weakly interact, such that the contribution of the noncondensed atoms to the photon flux is small and can be neglected. In this limit the chemical potential of both condensates is equal and given by $\mu(0)$. With definition (32) then $\mu_R = \mu(0)$ and $\mu_L = \mu(0) + \Delta V$.

The photon flux is evaluated using the Thomas-Fermi approximation for the condensate wave functions [29],

$$f_j(\mathbf{r}) = [(\mu(0) - V(\mathbf{r} - \mathbf{r}_j))/g]^{1/2}, \quad (33)$$

where $\mu(0) = (15N_C a_s/\bar{a})^{2/5} \hbar\bar{\omega}/2$ is the chemical potential at zero temperature and $\bar{a} = \sqrt{\hbar/(m\bar{\omega})}$ is the size of the ground state of a harmonic oscillator with frequency $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. The condensate macroscopic wave function has size $r_{\{\ell=x,y,z\}}^{(0)} = \sqrt{2\mu(0)/(m\omega_\ell^2)}$. Its Fourier transform reads $\tilde{f}_L(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{d}/2}\tilde{f}_0(\mathbf{k})$ (for the right condensate $\tilde{f}_R(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{d}/2}\tilde{f}_0(\mathbf{k})$) where $\tilde{f}_0(\mathbf{k})$ is the Fourier transform of the macroscopic wave function centered at the origin and is real valued. In particular

$$\tilde{f}_0(\mathbf{k}) = \kappa_0 \frac{|J_2(p_0)|}{p_0^2}, \quad (34)$$

where $\kappa_0 = \sqrt{15\pi^3 N_C r_x^{(0)} r_y^{(0)} r_z^{(0)}/2}$ is a scalar and $J_2(p_0)$ the Bessel function of second order for the variable p_0 defined as $p_0^2 = k_x^2 r_x^{(0)2} + k_y^2 r_y^{(0)2} + k_z^2 r_z^{(0)2}$, see [29]. The integral for the interference contribution to the photon flux in Eq. (26) can be now cast in the form

$$F_I(t) = 2\pi\kappa_0^2\Gamma \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{|J_2(p_0)|^2}{p_0^4} \text{Re} \left[e^{i(\delta\mu t - \mathbf{k}\cdot\mathbf{d} - \varphi_{LR})} \tilde{\delta}^t(\Omega - \omega_{\mathbf{k}+\mathbf{q}}) + e^{-i(\delta\mu t - \mathbf{k}\cdot\mathbf{d} - \varphi_{LR})} \tilde{\delta}^t(\Omega - \delta\mu - \omega_{\mathbf{k}+\mathbf{q}}) \right]. \quad (35)$$

This equation can be simplified taking that both \mathbf{d} and \mathbf{q} point along the positive x -direction. Following the derivation in App. B we find that the total flux can be approximated by the expression

$$F(t) \approx 2\pi\Gamma N_C K_m \left[1 + \mathcal{V}_0 \cos \left(\delta\mu t - \varphi_{LR} + (\omega_{\mathbf{q}} - \Omega) \frac{d}{v_q} \right) \Theta \left(t - \frac{d - 2r_x}{v_q} \right) \right], \quad (36)$$

where

$$K_m = (K(\omega_{\mathbf{q}} - \Omega) + K(\omega_{\mathbf{q}} - \Omega + \delta\mu))/2$$

and $K(x) = \sqrt{1/(2\pi\sigma^2)} \exp[-x^2/(2\sigma^2)]$ is a Gaussian of

width $\sigma^2 = 2.5(v_{\mathbf{q}}/r_x)^2$. Equation (36) shows that the photon flux starts oscillating for times $t > t_c$, with $t_c = (d - 2r_x)/v_{\mathbf{q}}$, namely, when the outcoupled atoms from one condensate have reached the second one. The time-

dependent oscillations have frequency $\delta\mu$ and amplitude

$$\mathcal{V}_0 = 2 \frac{K(\omega_{\mathbf{q}} - \Omega)}{K(\omega_{\mathbf{q}} - \Omega) + K(\omega_{\mathbf{q}} - \Omega + \delta\mu)}. \quad (37)$$

The expression in Eq. (36) has been obtained neglecting the momentum dependence of the trapped clouds (this approximation is verified in App. B) and the contribution of the noncondensed fraction. We note that, although oscillations as a function of time are observed provided that $\delta\mu \neq 0$, nevertheless their visibility (which corresponds to \mathcal{V}_0) is always smaller than unity, $\mathcal{V}_0 < 1$ for $\delta\mu \neq 0$. This can be understood considering that a scattered photon carries information about from which cloud it was scattered because of energy conservation in the scattering process.

We now turn our attention to the phase offset characterising the oscillations of the photon flux as a function of time in Eq. (36). This phase offset is here given by the quantity $\Theta - \varphi_{LR}$, where φ_{LR} is the relative phase between the two BEC, which can take any value (we refer to the discussion in Sec. III C), and by the quantity

$$\Theta = \frac{d}{v_q}(\omega_{\mathbf{q}} - \Omega + \delta\mu), \quad (38)$$

which is the phase the outcoupled atoms accumulate when travelling from left to the right well (The corresponding phase offset for the outcoupled atoms travelling from the right to the left well is given in App. B). This phase can be varied by either tuning the laser frequency, and thus Ω , or changing the distance between the minima of the two wells. The phase offset in Eq. (38) agrees with the one obtained in Ref. [31] that was derived from a phenomenological model [56].

We also note that our theory correctly predicts the oscillation period observed in the experiment and also reproduces the behaviour that the interference current needs some time to build up. The linear response treatment we apply, however, does not predict any decrease in the visibility of the interference pattern with time. This is in marked contrast with the experimental results. Possible reasons for the decrease in visibility in the experimental data are depletion of the condensate and heating of the sample, which could be due to spontaneous Rayleigh scattering events [30]. These effects are not taken into account in our model, but could be introduced by means of quantum Langevin equations using a formalism similar to the one developed in Ref. [35].

We finally discuss the dependence of the photon flux on time and on the angle of emission of the outcoupled atoms, thus for geometries where the direction of emission \mathbf{q} of the outcoupled atoms forms an angle α with the vector \mathbf{d} . For simplicity we assume that both vectors lie in the $x - y$ plane. The calculation is performed by numerically integrating Eq. (35).

Figure 2 displays the photon flux $F(t)$ as a function of time and of the emission angle α . Oscillations as a function of time are observed for values of the angle

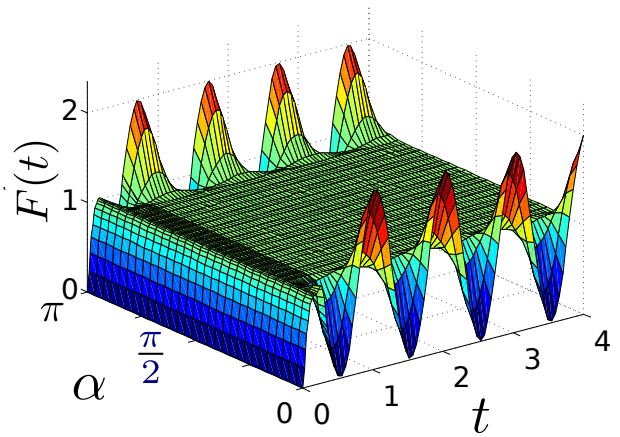


FIG. 2: (color online) Photon flux $F(t)$ (in units of the background contribution $F_B(t)$) as a function of time (in units of $\delta\mu/2\pi$) and of the angle α at which the atoms are emitted. The photon flux is computed by numerically integrating Eq. (35). The condensates are composed by $N = 10^6$ sodium atoms in a spherical harmonic trap of frequency $\omega = 325\text{Hz}$. The scattering length is $a_s = 55a_0$ with a_0 Bohr radius. The other parameters are $v_{\mathbf{q}} = 6 \frac{\text{cm}}{\text{s}}$, $\delta\mu_0 = 2\pi 10^3\text{Hz}$, $d = 5r_x$, $\Omega = \omega_{\mathbf{q}}$, $\alpha = 0$ and $\varphi_{LR} = 0$.

about $\alpha = 0, \pi$ corresponding to the atoms propagating in the direction parallel to $\pm \mathbf{d}$. The oscillation period is $2\pi/\delta\mu$ and is independent of the angle of emission. The oscillations disappear for angles in the interval $\pi/8 \lesssim \alpha \lesssim 7\pi/8$: the photon flux is here solely given by the background contribution $F_B(t)$. For these angles, in fact, at all times there is no spatial overlap between the wavefunction of the outcoupled atoms and the wavefunction of the trapped atoms in the other well. Indeed, from a simple geometric argument one finds that the overlap vanishes for angle $\alpha > \arctan \frac{2r_x}{d} \approx \pi/8$ and $\alpha < \pi - \pi/8$. This implies that the setup must be so constructed that the atoms outcoupled from one well could in principle be transferred, by a Raman process, into the second well. This property is the key element on which the analogy to a Josephson Junction has been drawn [31]. It is also basically the way in which a quantum eraser is realised in this setup. We refer the reader to the discussion on the properties of this quantum interference process in Sec. III C.

B. Interference between two Bose-Einstein condensates at different temperatures

We now show how the setup of Ref. [30] could be used to determine the condensate fraction of a Bose-Einstein condensate at a different temperature, using as reference a second BEC at known temperature. For this purpose, we make the same assumptions as in the previous section, with the difference that while the left gas is a BEC at temperature $T = 0$, the gas in the right well is a BEC at temperature T . We further assume that the Thomas-

Fermi approximation can be performed also for the right condensate. Hence, the chemical potential of the second condensate can be written as $\mu(T) = \mu(0)(1 - T^3/T_c^3)^{2/5}$, with T_c the critical temperature for the noninteracting gas in a harmonic trap, while the size of the macroscopic

wave function of the right condensate scales with $r_\ell^{(T)} = r_\ell^{(0)}(1 - T^3/T_c^3)^{1/5}$, see [29]. Using these relations the integral for the interference contribution to the photon flux can be cast in the form

$$F_I(t) = 2\pi\Gamma\kappa_0^2 n_C(T) \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{|J_2(p_0)J_2(p_T)|}{p_T^4} \times \text{Re} \left[e^{i(\delta\mu(T)t - \mathbf{k} \cdot \mathbf{d} - \varphi_{LR})} \tilde{\delta}^t(\Omega - \omega_{\mathbf{k}+\mathbf{q}}) + e^{-i(\delta\mu(T)t - \mathbf{k} \cdot \mathbf{d} - \varphi_{LR})} \tilde{\delta}^t(\Omega - \delta\mu(T) - \omega_{\mathbf{k}+\mathbf{q}}) \right], \quad (39)$$

with $p_T = (1 - T^3/T_c^3)^{1/5} p_0$. The integral depends on the temperature both through a scaling factor as well as the function $J_2(p_T)/p_T^4$ in the integrand, while the phases depend on the temperature of the second BEC via the chemical potential of the right BEC which enters in the quantity

$$\delta\mu(T) = \delta\mu(0) - \mathcal{F}(N, T)/\hbar, \quad (40)$$

with $\mathcal{F}(N, T) = \mu(T) - \mu(0)$. We note that $F_I(t)$ oscillates as a function of time with both frequency and amplitude which depend on the temperature of the second condensate. For $T \ll T_c$, in particular, the integral in Eq. (39) delivers the factor $1/\sqrt{n_C(T)}$, and thus $F_I(t) \propto \sqrt{n_C(T)}$, where $n_C(T) = N_C(T)/N = 1 - (T/T_c)^3$ is the condensate fraction in the right trap.

Figure (3a) displays $F_I(t)$ as function of time and temperature T of the right condensate when the trap is a spherical harmonic oscillator. The oscillations are visible for times $t > t_c$. The oscillation frequency depends on T , as one can clearly observe from the figure [57]. The dependence of the amplitude of the oscillation on the temperature is visible in Fig. (3a) and is singled out in Fig. (3b) where the amplitude evaluated at times $t > t_c$,

$$\mathcal{C}(T) = [\max(F_I(t)) - \min(F_I(t))], \quad (41)$$

is displayed as a function of T in units of $\mathcal{C}(0)$. The red dashed curve represents the squared root of the condensate fraction, $\sqrt{n_C(T)}$, and is shown for comparison. Function $\mathcal{C}(T)$ decreases monotonically as the temperature increases from $T = 0$, and vanishes at the critical temperature $T = T_c^i$ (where the superscript i indicates that we take into account the effect of the interactions and T_c^i is determined from the condition $n_C(T_c^i) = 0$) [58].

V. FIRST-ORDER CORRELATION FUNCTION

Several methods that are based on time-of-flight techniques for determining the first-order correlation functions have been proposed in the literature, see for in-

stance Refs. [3, 59, 60]. Measurements of the first-order correlation function have been performed on Bose-Einstein Condensates in Refs. [61, 62].

In the following we will show how an extension of our previously considered setup may allow one to determine the spatial first-order correlation function of a quantum gas by the scattered photons. In this case, the lasers shall illuminate two spatially separated regions of a quantum gas confined in a *single well* potential. More specifically, the spatial dependence of the laser-atom interaction in Eq. 3 will be characterized by the Gaussian envelope

$$|\gamma_j(\mathbf{r})| = \gamma_0 \exp(-(\mathbf{r} - \mathbf{r}_j)^2 / \Delta \mathbf{r}^2) / (\sqrt{\pi} \Delta \mathbf{r})^{1/2},$$

with width $\Delta \mathbf{r} \ll |\mathbf{d}|$ between the two regions (We remark that the excitation could be realized with subwavelength resolution [6, 7, 63]). The corresponding setup is shown in Fig. 4. For this setup the photon flux reads

$$F(t) = \Gamma \sum_{j,k=L,R} \text{Re} \int_0^t dt' \int d\mathbf{r} \int d\mathbf{r}' \times \gamma_j(\mathbf{r}) \gamma_k(\mathbf{r}') f(\mathbf{r}, t; \mathbf{r}', t') G^{(1)}(\mathbf{r}, t; \mathbf{r}', t') \quad (42)$$

where

$$G^{(1)}(\mathbf{r}, t; \mathbf{r}', t') = \langle \psi_1^\dagger(\mathbf{r}, t) \psi_1(\mathbf{r}', t') \rangle \quad (43)$$

is the first-order correlation function and $f(\mathbf{r}, t; \mathbf{r}', t')$ is defined in Eq. (15). Let the pump-probe excitation be a pulse of mean duration t such that $\tilde{\omega}t \gg 1$, but $\omega_\alpha t \ll 1$, with ω_α the typical frequency characterising the excitation spectrum. In this limit we can approximate $G^{(1)}(\mathbf{r}, t; \mathbf{r}', t') \simeq G^{(1)}(\mathbf{r}, 0; \mathbf{r}', 0)$ in Eq. (42). For convenience we denote by $G^{(1)}(\mathbf{r}; \mathbf{r}') \equiv G^{(1)}(\mathbf{r}, 0; \mathbf{r}', 0)$ the spatial correlation function. For $\Delta \mathbf{r}$ sufficiently small, so that it can be approximated with a δ -like excitation, the photon flux can be recast in the form

$$F(t) \simeq K \left(1 + \text{Re} \left\{ e^{i\mathbf{q} \cdot \mathbf{d}} G^{(1)}(\mathbf{d}/2; -\mathbf{d}/2) \right\} / n_0(\mathbf{d}/2) \right) \quad (44)$$

where K is a constant which is determined by the details of the excitation scheme and $n_0(\mathbf{d}/2) = G^{(1)}(\mathbf{d}/2; \mathbf{d}/2)$

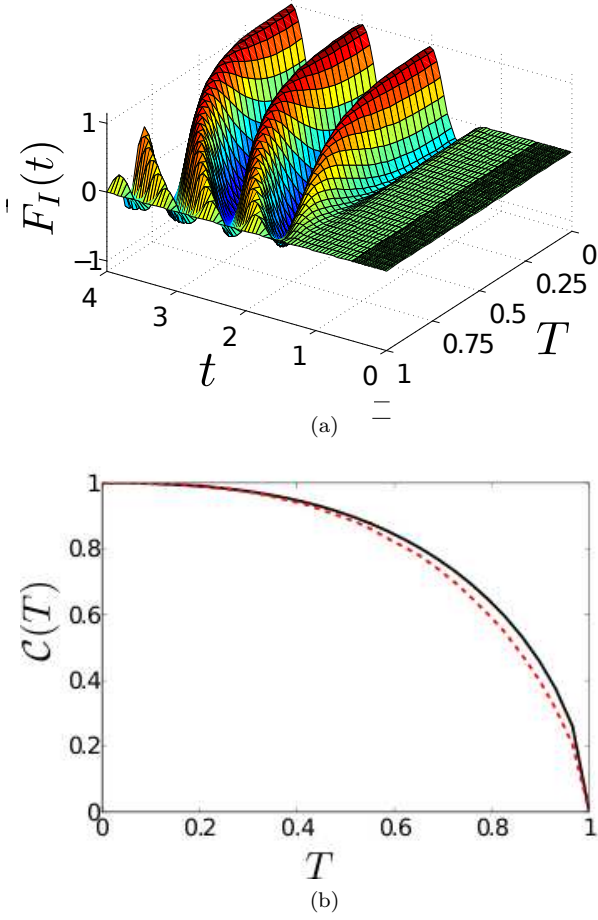


FIG. 3: (color online) (a) Interference term of the Photon flux, $\bar{F}_I(t)$, Eq. (39), (in units of the background current obtained when both condensates are at zero temperature $F_B(T=0)$), as a function of time (in units of $2\pi/\delta\mu$) and temperature T (in units of critical temperature T_c^i). (b) Amplitude $\mathcal{C}(T)$, Eq. (41) (in units of $\mathcal{C}(0)$) as a function of the temperature T (in units of T_c^i). The red dashed line corresponds to the squared root of the condensate fraction in the right BEC, $\sqrt{n_C(T)}$. The other parameters are as in Fig. 2.

is the density at $\mathbf{r} = \pm \mathbf{d}/2$ (assuming the system has reflection symmetry about $\mathbf{r} = 0$). Therefore, the photon flux exhibits oscillations with a visibility which is determined by the spatial first-order correlation function. By varying the scattering wave vector \mathbf{q} one would thus measure the first-order correlation function as a function of \mathbf{d} . Realistic excitation schemes are of course characterized by finite spatial resolution $\Delta \mathbf{r}$. This results in averaging the correlation function in Eq. (44) over the finite size of the illuminated region and hence to a diminution of the contrast [55].

This setup could be extended to measure time-dependent correlation function by applying a pair of laser pulses: Assuming that the photon scattered after the first pulse can interfere with the photon scattered after the

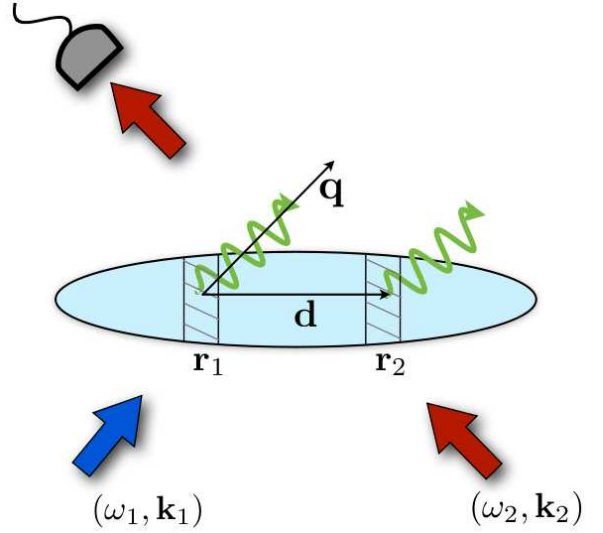


FIG. 4: Setup for measuring the first order correlation function of an ultracold atomic gas. The pump probe lasers are tailored in such a way that atoms are only outcoupled from the two shaded regions around $\mathbf{r}_{1,2}$. Measurement of the photon flux of one beam as a function of \mathbf{q} allows one to determine the first order correlation function $G^{(1)}(\mathbf{r}_1; \mathbf{r}_2)$ of the atomic gas.

second pulse, then the photon flux at the detector will exhibit oscillations whose amplitude is proportional to the first-order correlation function Eq.(43). A possible realization could use a mirror placed in front of the quantum system as realized in [64].

VI. CONCLUSIONS

In this work we have discussed a setup which allows one to measure the condensate fraction and the first-order correlation function of a quantum gas by means of photo-detection. The photons are scattered by the quantum gas in a pump-probe type of excitation, such that the scattered photon is associated with an outcoupled atom with which it is entangled. In addition, photon and atom are correlated with the quantum gas. This correlation is detected in the photon flux, provided that certain conditions are fulfilled which we have identified and discussed.

Our analysis is based on the impulse approximation [45] corresponding to neglecting the back action of the scattering process on the quantum gas. It is therefore valid for short time transients. From the point of view of collecting a sufficient statistics, hence, time-of-flight techniques are a more convenient tool than in-situ measurements by photodetection. Nevertheless, one could consider to modify existing techniques, such as the one successfully demonstrated in Refs. [6, 7], to access to the same kind of information that the setup here discussed provides.

An interesting outlook is to identify a setup, along the lines of the proposal Ref. [65], which permits one to perform a quantum-non-demolition measurement of any correlations function of the external degrees of freedom of atomic gases, and more in general, which realizes quantum-state transfer between matter and light. This would open several interesting perspectives for quantum communications [66].

Acknowledgments

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Appendix A

We recast the interference term, Eq. (13b), in the form $F_I = F_{L \rightarrow R}(t) + F_{R \rightarrow L}(t)$, with

$$\begin{aligned}
 F_{L \rightarrow R}(t) &= \Gamma \text{Re} e^{i\delta\mu t} \int_0^t dt' \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i(\Omega - \omega_{\mathbf{k}})(t-t')} \\
 &\quad \times \int d\mathbf{r} d\mathbf{r}' e^{i(\mathbf{q}-\mathbf{k}) \cdot (\mathbf{r}-\mathbf{r}')} \Phi_L(\mathbf{r}')^* \Phi_R(\mathbf{r}) \quad (\text{A1a}) \\
 F_{R \rightarrow L}(t) &= \Gamma \text{Re} e^{-i\delta\mu t} \int_0^t dt' \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i(\Omega - \omega_{\mathbf{k}} - \delta\mu)(t-t')} \\
 &\quad \times \int d\mathbf{r} d\mathbf{r}' e^{i(\mathbf{q}-\mathbf{k}) \cdot (\mathbf{r}-\mathbf{r}')} \Phi_R(\mathbf{r}')^* \Phi_L(\mathbf{r}) . \quad (\text{A1b})
 \end{aligned}$$

$$F_{L \rightarrow R}(t) = \Gamma \text{Re} e^{i(\delta\mu t - \varphi_{LR})} \int_0^t d\tau \left(\frac{i m}{2\pi\hbar\tau} \right)^{\frac{3}{2}} \int d\mathbf{r} \int d\mathbf{r}' e^{i(\mathbf{q} \cdot (\mathbf{r}-\mathbf{r}') - \Omega\tau)} \exp \left[-\frac{i}{\hbar} \frac{m(\mathbf{r}' - \mathbf{r})^2}{2\tau} \right] f_L(\mathbf{r}') f_R(\mathbf{r}) , \quad (\text{A2})$$

where φ_{LR} is the relative phase of the macroscopic wavefunctions defined in Eq. (28). With the change of variables $\bar{\mathbf{r}} = \mathbf{r} - \mathbf{r}'$, $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$, we can rewrite Eq. (A2) as

$$F_{L \rightarrow R}(t) = \Gamma \text{Re} e^{i(\delta\mu t - \varphi_{LR})} \int_0^t d\tau \left(\frac{i m}{2\pi\hbar\tau} \right)^{\frac{3}{2}} e^{i(\omega_{\mathbf{q}} - \Omega)\tau} \int d\bar{\mathbf{r}} \int d\mathbf{R} \exp \left[\frac{i}{\hbar} \frac{m}{2\tau} \left(\bar{\mathbf{r}} - \frac{\hbar\mathbf{q}}{m}\tau \right)^2 \right] f_L(\mathbf{R} - \frac{\bar{\mathbf{r}}}{2}) f_R(\mathbf{R} + \frac{\bar{\mathbf{r}}}{2}) \quad (\text{A3})$$

The exponential in the integral over $\bar{\mathbf{r}}$ oscillates very fast with respect to the wave functions $f_j(\mathbf{r})$. Therefore

We now show that the term $F_{L \rightarrow R}(t)$ ($F_{R \rightarrow L}(t)$) is the contribution due to the atoms which are outcoupled and propagate from the left to the right (right to the left). For this purpose we perform the \mathbf{k} integral in Eq. (A1) and obtain

the main contribution to the integral over $\bar{\mathbf{r}}$ comes from

$\bar{\mathbf{r}}_0 = \frac{\hbar \mathbf{q}}{m} \tau$, where the term in the exponential vanishes. By means of the saddle-point approximation we take the wave functions at the point $\bar{\mathbf{r}}_0$ out of the integral. Performing the $\bar{\mathbf{r}}$ integration using the Fresnel integral [28]

$$\int_{-\infty}^{\infty} dt e^{i\gamma t^2} = \sqrt{\frac{\pi}{|\gamma|}} e^{i \text{sign}(\gamma) \pi/4}, \quad (\text{A4})$$

we obtain

$$F_{L \rightarrow R}(t) \approx \Gamma \text{Re} e^{i(\delta\mu t - \varphi_{LR})} \int_0^t d\tau e^{i(\omega_{\mathbf{q}} - \Omega)\tau} \times \int d\mathbf{r} f_L(\mathbf{r} + \mathbf{d} - v_{\mathbf{q}}\tau) f_R(\mathbf{r}), \quad (\text{A5})$$

where $v_{\mathbf{q}}$ is the recoil velocity, Eq. (31). The calculation that leads to Eq. (A5) is exact in the limit of homogeneous atomic systems. In such case the momentum distribution of the condensate fraction is a Dirac-delta function at zero momentum and all outcoupled atoms have exactly the same momentum $\hbar \mathbf{q}$. The condensates we consider are confined by an external potential and have a finite extension in space which leads to a certain width δp in their momentum distribution, and thus to a spread in the momentum of the outcoupled atoms around the mean value $\hbar \mathbf{q}$. By taking the value of the atomic wave functions at the point of the stationary phase of the exponential in Eq. (A3) one neglects this momentum width: The saddle-point approximation is thus applicable if $\hbar |\mathbf{q}| \gg \delta p$. Equation (A5) agrees with the corresponding expression in Eq. (29a). For completeness we also give the contributions of the macroscopic wavefunctions to the background contribution, Eq. (13a), making the same approximations as in Eqs. (A5):

$$F_L(t) \approx \Gamma \text{Re} \int_0^t d\tau e^{i(\omega_{\mathbf{q}} - \Omega)\tau} \int d\mathbf{r} f_L(\mathbf{r}) f_L(\mathbf{r} - v_{\mathbf{q}}\tau),$$

$$F_R(t) \approx \Gamma \text{Re} \int_0^t d\tau e^{i(\omega_{\mathbf{q}} - \Omega + \delta\mu)\tau} \int d\mathbf{r} f_R(\mathbf{r}) f_R(\mathbf{r} - v_{\mathbf{q}}\tau). \quad (\text{A6})$$

Appendix B

We derive here approximate expressions of the Raman scattering rate for the experimental setup of [30], which lead to Eq. (36). Using Eq. (33) for the condensate wavefunctions in Eq. (A6) we find

$$F_L(t) \approx \frac{2\pi\Gamma\mu(0)r_x\bar{R}^3}{g_1 v_{\mathbf{q}}} \int_0^{z_<(0)} dz \cos\left[(\omega_{\mathbf{q}} - \Omega) \frac{z r_x}{v_{\mathbf{q}}}\right] G(z), \quad (\text{B1})$$

with

$$z_<(x) = \min(2, \frac{v_{\mathbf{q}} t - x}{r_x}). \quad (\text{B2})$$

The length $\bar{R} = (r_x r_y r_z)^{1/3}$ determines the typical size of the condensates and can be written as [67]

$$\bar{R} = 15^{1/5} \left(\frac{N a_s}{\bar{a}}\right)^{1/5} \bar{a}. \quad (\text{B3})$$

The function $G(z)$ in Eq. (B1) reads

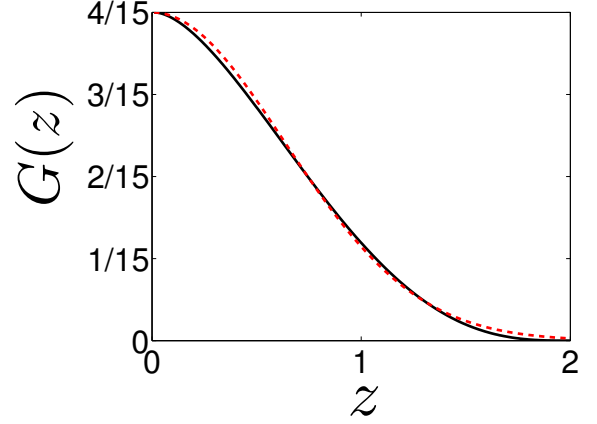


FIG. 5: (color online) Comparison of the integral $G(z)$ as given in Eq. (B4) (black solid line) with the Gaussian fit Eq. (B5) (red dashed line).

$$G(z) = 2 \int_{z/2}^1 dx \int_0^{\sqrt{1-x^2}} r dr (1 - x^2 - r^2)^{1/2} \times (1 - (x - z)^2 - r^2)^{1/2}. \quad (\text{B4})$$

A numerical evaluation of $G(z)$ is shown in Fig. (5), and is here compared with a Gaussian of the form

$$G(z) \approx \frac{4}{15} e^{-1.25 z^2}, \quad (\text{B5})$$

showing that this function provides a good approximation of Eq. (B4). Using Eq. (B5) in Eq. (B1) one gets

$$F_L(t) \approx \Gamma N_C t_0 \int_0^{z_<(0)} dz \cos[(\omega_{\mathbf{q}} - \Omega) z t_0] e^{-1.25 z^2}, \quad (\text{B6})$$

with $t_0 = r_x/v_{\mathbf{q}}$. For times $t > t_c$, with $t_c \sim (d - 2r_x)/v_{\mathbf{q}}$, such that $z_<(0) = z_<(d) = 2$ we find from Eq. (B6)

$$F_L(t) \approx \Gamma t_0 N_C \int_0^2 dz \cos\left[(\omega_{\mathbf{q}} - \Omega) \frac{z r_x}{v_{\mathbf{q}}}\right] e^{-1.25 z^2} \approx \pi \Gamma N_C \sqrt{\frac{t_0^2}{5\pi}} e^{-\frac{1}{5}[(\omega_{\mathbf{q}} - \Omega) t_0]^2}. \quad (\text{B7})$$

When performing the integral in Eq. (B7) we neglected the imaginary part in the Error-functions which one gets from the exact integration. This is a good approximation for $|(\omega_{\mathbf{q}} - \Omega)t_0| \leq 5$, as can be checked by numerical evaluation. For values $|(\omega_{\mathbf{q}} - \Omega)t_0| > 5$ the exponential in Eq. (B7) is negligible compared to the resonant case $\omega_{\mathbf{q}} = \Omega$: The oscillating tails of the Error-functions given by their imaginary parts will only play a role for parameters where the outcoupling efficiency vanishes and which are thus not relevant to our treatment. Using the same argumentation for the other contributions to the photon flux we find

$$F_L(t) \approx \pi \Gamma N_C K(\omega_{\mathbf{q}} - \Omega), \quad (\text{B8a})$$

$$F_R(t) \approx \pi \Gamma N_C K(\omega_{\mathbf{q}} - \Omega + \delta\mu), \quad (\text{B8b})$$

$$F_{L \rightarrow R}(t) \approx 2\pi \Gamma N_C K(\omega_{\mathbf{q}} - \Omega) F_{\Theta}(q, t) \times \cos \left[\delta\mu t - \varphi_{LR} + (\omega_{\mathbf{q}} - \Omega) \frac{d}{v_q} \right], \quad (\text{B8c})$$

$$F_{R \rightarrow L}(t) \approx 2\pi \Gamma N_C K(\omega_{\mathbf{q}} - \Omega + \delta\mu) F_{\Theta}(-q, t) \times \cos \left[\delta\mu t - \varphi_{LR} - (\omega_{\mathbf{q}} - \Omega + \delta\mu) \frac{d}{v_q} \right], \quad (\text{B8d})$$

with

$$K(x) = \sqrt{\frac{t_0^2}{5\pi}} e^{-\frac{t_0^2}{5} x^2}, \quad (\text{B9})$$

$$F_{\Theta}(q, t) = \Theta(q) \Theta \left(t - \frac{d - 2r_x}{v_q} \right). \quad (\text{B10})$$

From Eqs. (B8) one then obtains Eq. (36) for $q > 0$. The phase difference between $F_{L \rightarrow R}(t)$ and $F_{R \rightarrow L}(t)$ as measured in [31] is obtained from Eqs. (B8c,d) and reads

$$\Theta = \frac{d}{v_q} (2\Omega - 2\omega_{\mathbf{q}} - \delta\mu). \quad (\text{B11})$$

Figure (6) displays the total photon flux computed from Eq. (36) (dashed line) and the one calculated from Eq. (35) (solid line) for the experimental parameters of [30]. Quantitative agreement between the two solutions is found, showing that neglecting the initial momentum distribution is well justified for the considered parameters.

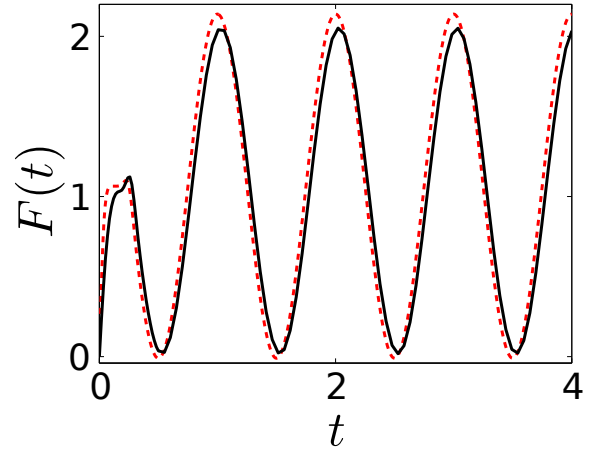


FIG. 6: (color online) Total photon Flux $F(t)$ in units of background contribution $F_B(t)$ as a function of time in units of $\frac{d}{2\pi}$. The solid black line is computed by numerically integrating Eq. (35) and compared to the approximate result Eq. (36) (dashed red line). Parameters are as in Fig. (2) with $\alpha = 0$.

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- [1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999)
 - [2] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
 - [3] A. Imambekov, V. Gritsev, and E. Demler, "Fundamental noise in matter interferometers", in "Ultracold Fermi gases", Proceedings of the International School of Physics Enrico Fermi, 2006 (IOS Press, Amsterdam, The Netherlands, 2007).
 - [4] R. Bach and K. Rzażewski, Phys. Rev. Lett. **92**, 200401 (2004)
 - [5] S. P. Rath and W. Zwerger, Phys. Rev. A **82**, 053622 (2010).
 - [6] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. I. Gillen, S. Fölling, L. Pollet, and M. Greiner, Science **329**, 547 (2010).
 - [7] J. F. Sherson, C. Weitenberg, M. Endres, M. Cheneau, I. Bloch, and S. Kuhr, Nature **467**, 68 (2010); C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schauß, T. Fukuhara, I. Bloch, and S. Kuhr, Nature **471**, 319 (2011).
 - [8] R. Ma, M. E. Tai, P. M. Preiss, W. S. Bakr, J. Simon, and M. Greiner, Phys. Rev. Lett. **107**, 095301 (2011)
 - [9] M. Endres, M. Cheneau, T. Fukuhara, C. Weitenberg, P. Schauß, C. Gross, L. Mazza, M. C. Bañuls, L. Pollet, I. Bloch, and S. Kuhr, Science **334**, 200 (2011).
 - [10] Y. Colombe, T. Steinmetz, G. Dubois, F. Linke, D. Hunger, and J. Reichel, Nature **450**, 272 (2007).
 - [11] K. Baumann, C. Guerlin, F. Brennecke and T. Esslinger, Nature **464**, 1301 (2010); K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. **107**, 140402 (2011).
 - [12] S. Bux, C. Gnahn, R. A. W. Maier, C. Zimmermann, and

- Ph. W. Courteille, Phys. Rev. Lett. **106**, 203601 (2011).
- [13] K. W. Murch, K. L. Moore, S. Gupta, and D. M. Stamper-Kurn, Nature Phys. **4**, 561 (2008); N. Brahms, T.P. Purdy, D.W.C. Brooks, T. Botter, and D.M. Stamper-Kurn, preprint arXiv:1012.1285 (2010), to appear in Nature Phys. (2011).
- [14] W. Chen, D. Meiser, and P. Meystre, Phys. Rev. A **75**,

- 023812 (2007).
- [15] I. B. Mekhov, C. Maschler and H. Ritsch, *Nature Phys.* **3**, 319 (2007).
- [16] J. Larson, B. Damski, G. Morigi, and M. Lewenstein, *Phys. Rev. Lett.* **100**, 050401 (2008).
- [17] I. B. Mekhov and H. Ritsch, *Phys. Rev. Lett.* **102**, 020403 (2009).
- [18] M. Weidemüller, A. Hemmerich, A. Görlitz, T. Esslinger, and T.W. Hänsch, *Phys. Rev. Lett.* **75**, 4583 (1995); M. Weidemüller, A. Görlitz, T.W. Hänsch, and A. Hemmerich, *Phys. Rev. A* **58**, 4647 (1998).
- [19] G. Birkel, M. Gatzke, I. H. Deutsch, S. L. Rolston, and W. D. Phillips, *Phys. Rev. Lett.* **75**, 2823 (1995).
- [20] S. Slama, C. von Cube, A. Ludewig, M. Kohler, C. Zimmermann, and Ph. W. Courteille, *Phys. Rev. A* **72**, 031402(R) (2005); S. Slama, C. von Cube, B. Deh, A. Ludewig, C. Zimmermann, and Ph.W. Courteille, *Phys. Rev. Lett.* **94**, 193901 (2005); S. Slama, C. von Cube, M. Kohler, C. Zimmermann, and Ph. W. Courteille, *Phys. Rev. A* **73**, 023424 (2006).
- [21] C. Weitenberg, P. Schauf, T. Fukuhara, M. Cheneau, M. Endres, I. Bloch and S. Kuhr, *Phys. Rev. Lett.* **106**, 215301 (2011).
- [22] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, *Rev. Mod. Phys.* **77**, 187 (2005).
- [23] S. Rist, C. Menotti, and G. Morigi, *Phys. Rev. A* **81**, 013404 (2010).
- [24] J. S. Douglas and K. Burnett, *Phys. Rev. A* **84**, 033637 (2011).
- [25] M. Lewenstein and L. You, *Phys. Rev. Lett.* **71**, 1339 (1993).
- [26] J. Javanainen and J. Ruostekoski, *Phys. Rev. Lett.* **91**, 150404 (2003); J. Ruostekoski, J. Javanainen, and G.V. Dunne, *Phys. Rev. A* **77**, 013603 (2008).
- [27] J. Ruostekoski, C.J. Foot, and A.B. Deb, *Phys. Rev. Lett.* **103**, 170404(2009).
- [28] W. P. Schleich, *Quantum Optics in Phase Space*, Wiley ed. (New York, 2001).
- [29] L.Pitaevski and S. Stringari, *Bose-Einstein Condensation* (Oxford Science Publications, Oxford, 2003).
- [30] M. Saba, T. A. Pasquini, C. Sanner, Y. Shin, W. Ketterle and D. E. Pritchard, *Science* **307**, 1945 (2005).
- [31] Y. Shin, G.-B. Jo, M. Saba, T. A. Pasquini, W. Ketterle and D. E. Pritchard, *Phys. Rev. Lett.* **95**, 170402 (2005).
- [32] M. O. Scully and K. Drühl, *Phys. Rev. A* **25**, 2208 (1982).
- [33] M. Lewenstein, L. You, J. Cooper, and K. Burnett, *Phys. Rev. A* **50**, 2207 (1994).
- [34] S. Zippilli, B. Mohring, E. Lutz, G. Morigi, and W. P. Schleich, *Phys. Rev. A* **83**, 051602(R) (2011).
- [35] J. Larson, S. Fernandez-Vidal, G. Morigi, and M. Lewenstein, *New J. Phys.* **10**, 045002 (2008).
- [36] This interaction could be realized by means of a laser, pumping the atoms, and a cavity mode, acting as a probe, that are both far-detuned from the atomic excited state but are set close to resonance with an atomic Raman transition. A similar setup has been proposed in [15].
- [37] This commutation relation is well defined provided that $\mathbf{r}, \mathbf{r}' \neq 0$. The mathematical subtlety related to this specific point is irrelevant to our study, being practically zero the probability that an atom in the electronic state [1] can be found around this point in space.
- [38] D. L. Luxat and A. Griffin, *Phys. Rev. A* **65**, 043618 (2002).
- [39] S. Choi and Y. Japha and K. Burnett, *Phys. Rev. A* **61**, 063606 (2000).
- [40] It is interesting to note that in the considered setup this shift contains also the operator $\psi_L^{(0)\dagger}(\mathbf{r}, \tau)\psi_R^{(0)}(\mathbf{r}, \tau)$ and its adjoint. Hence, its measurement would reveal tunneling events between the wells. This is however not relevant for the case considered in this work, since we assume that there is no tunneling between the two separate wells.
- [41] We note that the interaction between the trapped and outcoupled atoms can be made small by means of a Feshbach resonance [2].
- [42] C. Menotti and N. Trivedi, *Phys. Rev. B* **77**, 235120 (2008).
- [43] The corresponding expressions for the right cloud are obtained by replacing $\Omega \rightarrow \Omega - \delta\mu$ where $\delta\mu = \frac{\mu_L - \mu_R}{\hbar}$.
- [44] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (Wiley, New York, 2004).
- [45] L. Pitaevskii and S. Stringari, *Phys. Rev. Lett.* **83**, 4237 (1999).
- [46] J. Ruostekoski and D. F. Walls, *Phys. Rev. A* **56**, 2996 (1997).
- [47] Z. Hadzibabic, S. Stock, B. Battelier, V. Bretin, and J. Dalibard, *Phys. Rev. Lett.* **93**, 180403 (2004).
- [48] S. Hofferberth, I. Lesanovsky, T. Schumm, A. Imambekov, V. Gritsev, E. Demler, and J. Schmiedmayer, *Nature Physics* **4**, 489 (2008); A. Imambekov, I. E. Mazets, D. S. Petrov, V. Gritsev, S. Manz, S. Hofferberth, T. Schumm, E. Demler, and J. Schmiedmayer, *Phys. Rev. A* **80**, 033604 (2009).
- [49] M. Iazzi and K. Yuasa, *Phys. Rev. A* **83**, 033611 (2011).
- [50] J. Javanainen and S. M. Yoo, *Phys. Rev. Lett.* **76**, 161 (1996), Y. Castin and J. Dalibard *Phys. Rev. A* **55**, 4330 (1997).
- [51] G. Magyar and L. Mandel, *Nature (London)* **198**, 255 (1963); R. L. Pflieger and L. Mandel, *Phys. Rev.* **159**, 1084 (1967).
- [52] H. Paul, *Rev. Mod. Phys.* **58**, 209 (1986).
- [53] B. G. Englert, *Phys. Rev. Lett.* **77**, 2154 (1996).
- [54] L.-M. Duan and C. Monroe, *Rev. Mod. Phys.* **82**, 1209 (2010).
- [55] P. Cañizares, T. Görler, J.P. Paz, G. Morigi, and W.P. Schleich, *Laser Physics* **17**, 903 (2007).
- [56] In Ref. [31], in particular, the rate of change of the difference in phase offset between atoms outcoupled to the left and atoms outcoupled to the right was measured as a function of the Bragg-frequency Ω , and found in agreement with the prediction of Eq. (B11) in App. (B).
- [57] We note that the condensate depletion, following from the outcoupling, induces a variation of N_C as a function of time, which also leads to a change in the chemical potential. This effect is not taken into account in our treatment, where we neglect the change of atom number in the condensates.
- [58] The critical temperature T_c^{emi} is determined from the condition $N_C(T = T_c^{\text{emi}}) = 0$ and is evaluated numerically.
- [59] L.-M. Duan, *Phys. Rev. Lett.* **96**, 103201 (2006).
- [60] T. L. Dao, A. Georges, J. Dalibard, C. Salomon and I. Carusotto, *Phys. Rev. Lett.* **98**, 240402 (2007).
- [61] I. Bloch, T. W. Hänsch and T. Esslinger, *Nature*, **403**, 6766 (2000).
- [62] S. Ritter, A. Öttl, T. Donner, T. Bourdel, M. Köhl and T. Esslinger, *Phys. Rev. Lett.* **98**, 090402 (2007).
- [63] A. V. Gorshkov, L. Jiang, M. Greiner, P. Zoller, and M. D. Lukin, *Phys. Rev. Lett.* **100**, 093005 (2008).

- [64] J. Eschner, Ch. Raab, F. Schmidt-Kaler, and R. Blatt, Nature (London) **413**, 495 (2001).
- [65] K. Ekert, O. Romero-Isart, M. Rodriguez, M. Lewenstein, E. Polzik and A. Sanpera, Nature Physics, **4** 50 (2008).
- [66] D. Chang, V. Gritsev. G. Morigi, V. Vuletic, M.D. Lukin, and E.A. Demler, Nature physics **4**, 884 (2008).
- [67] C. J. Pethick and H. Smith *Bose Einstein Condensation in Dilute Gases* (Cambridge University Press, 2008)